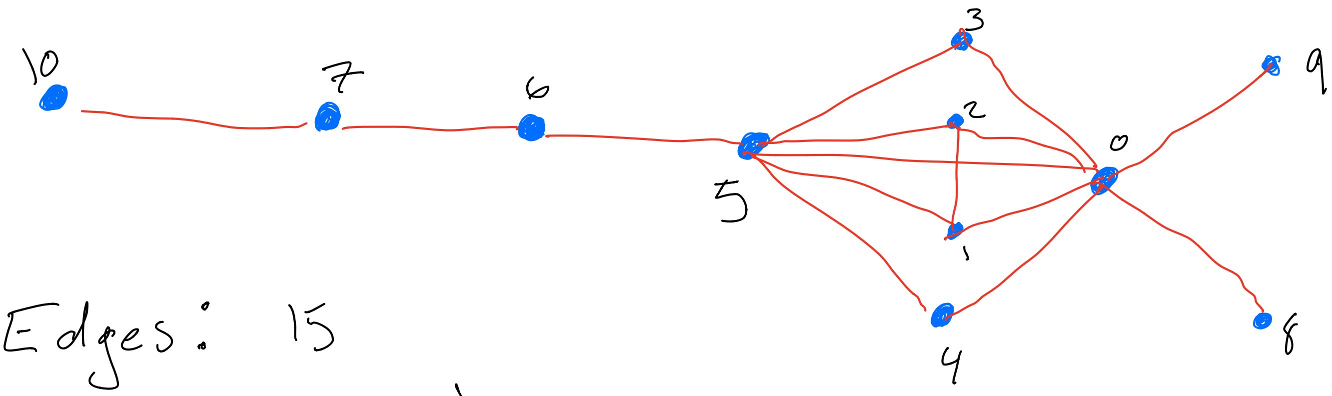

Edges: 15

Max Edges: $\frac{n(n-1)}{2}$ & $n = 11$

$$= \frac{11(10)}{2} = 55$$

Density $= \frac{15}{55} = \boxed{0.2727}$

Max Degree: $\boxed{\text{degree}(v_0) = 7}$

$\text{degree}(v_5) = 6$

$\text{degree}(v_{10}) = 1$

Adjacency Matrix:

$$A = \begin{array}{c|ccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
5 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}$$

$$k = \begin{bmatrix} 7 & 3 & 3 & 2 & 2 & 6 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$$

$\sum_i k_i = 30$, Edge Count: $\sum k/2 = 15$

Degree Centrality for:

$v_0 \Rightarrow \text{degree}(v_0) = c_0^d = \sum_{j=0}^{n} a_{0j} = 7$

$v_1 \Rightarrow \text{degree}(v_1) = c_1^d = \sum a_{1j} = 3$

$v_5 \Rightarrow \text{degree}(v_5) = c_5^d = \sum a_{5j} = 6$

$v_6 \Rightarrow \text{degree}(v_6) = c_6^d = \sum a_{6j} = 2$

$v_9 \Rightarrow \text{degree}(v_9) = c_9^d = \sum a_{9j} = 1$

Normalized Degree Centrality:

$\bar{c}_i^d = c_i^d / n-1$, $\boxed{\max_i \bar{c}_i^d = c_0^d / 11-1 = 7/10 = 0.7}$

Closeness Centrality:

also called the "geodesic"

$v_0 \Rightarrow \text{farness}(v_0) = \sum_{j \neq 0} \text{shortest\_distance}(v_0, v_j) =$

Edges on path — shortest path length

$\text{shortest\_distance}(v_0, v_1) + \leftarrow |(v_0, v_1)| = 1$

$\text{shortest\_distance}(v_0, v_2) + \leftarrow |(v_0, v_2)| = 1$

$\text{shortest\_distance}(v_0, v_3) + \leftarrow |(v_0, v_3)| = 1$

$\text{shortest\_distance}(v_0, v_4) + \leftarrow |(v_0, v_4)| = 1$

$\text{shortest\_distance}(v_0, v_5) + \leftarrow |(v_0, v_5)| = 1$

$\text{shortest\_distance}(v_0, v_6) + \leftarrow |(v_0, v_5), (v_5, v_6)| = 2$

$\text{shortest\_distance}(v_0, v_7) + \leftarrow |(v_0, v_5), (v_5, v_6), (v_6, v_7)| = 3$

$\text{shortest\_distance}(v_0, v_8) + \leftarrow |(v_0, v_8)| = 1$

$\text{shortest\_distance}(v_0, v_9) + \leftarrow |(v_0, v_9)| = 1$

$\text{shortest\_distance}(v_0, v_{10}) + \leftarrow |(v_0, v_5), (v_5, v_6), (v_6, v_7), (v_7, v_{10})| = 4$

Repeat for all v_j in graph. Omitted here for space & hand preservation.

$= 1 + 1 + 1 + 1 + 1 + 2 + 3 + 1 + 1 + 4 = 16$

$\text{Closeness}(v_0) = 1/\text{farness}(v_0) = 1/16$

$v_5 \Rightarrow \text{farness}(v_5) = \sum_{j \neq 5} \text{shortest\_distance}(v_5, v_j)$

$= 1 + 1 + 1 + 1 + 1 + 1 + 2 + 2 + 2 + 3 = 15$

$\text{closeness}(v_5) = 1/\text{farness}(v_5) = 1/15$

Normalized Closeness:

$\overline{\text{closeness}}(v_0) = (n-1) \cdot \text{closeness}(v_0) = 10/16 = 5/8 = \boxed{0.625}$

$\overline{\text{closeness}}(v_5) = (n-1) \cdot \text{closeness}(v_5) = 10/15 = 2/3 = \boxed{0.6667}$

Max closeness centrality @ v_5

Betweenness Centrality:

$$Btwn(v_0) = \sum_{j,k \neq 0} \frac{\sigma_{jk}(v_0)}{\sigma_{jk}} \ni$$

σ_{jk} is the # of shortest paths between v_j & v_k.

$\sigma_{jk}(v_0)$ is # of shortest paths between v_j & v_k going through v_0.

First need all pairs of nodes excluding v_0:

(v_1,v_2), (v_1,v_3), (v_1,v_4), (v_1,v_5), (v_1,v_6), (v_1,v_7), (v_1,v_8), (v_1,v_9), (v_1,v_{10})

(v_2,v_3), (v_2,v_4), (v_2,v_5), (v_2,v_6), (v_2,v_7), (v_2,v_8), (v_2,v_9), (v_2,v_{10})

$\vdots$ etc.

$\sigma_{12} \leftarrow$ how many shortest paths are there between v_1 & v_2?

Answer: Just 1 since they are adjacent.

$\sigma_{12} = 1$

$\sigma_{12}(v_0) \leftarrow$ how many shortest paths btwn v_1 & v_2 go through v_0?

Answer: None

$\sigma_{12}(v_0) = 0$

$$\therefore \frac{\sigma_{12}(v_0)}{\sigma_{12}} = 0/1 = 0$$

Repeat for all pairs. Example continues w/ $\dfrac{\sigma_{58}(v_0)}{\sigma_{58}}$

$$\sigma_{58} = \left|\left((v_5, v_0), (v_0, v_8)\right)\right| = 1$$

Note: $v_5 \Rightarrow v_1 \Rightarrow v_0 \Rightarrow v_8$ is a path but not the shortest path

$\sigma_{58}(v_0) = 1$

$$\therefore \frac{\sigma_{58}(v_0)}{\sigma_{58}} = 1/1 = 1$$

You can build a matrix here for σ_{ij} to count shortest paths

2 because two shortest paths exist: $((v_1,v_5),(v_5,v_3))$ & $((v_1,v_0),(v_0,v_3))$, both with length 2

$$\sigma = \begin{array}{c|ccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & & 0 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & & & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & & & & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & & & & & & 0 & 1 & 1 & 1 & 1 & 1 \\
6 & & & & & & & 0 & 1 & 1 & 1 & 1 \\
7 & & & & & & & & 0 & 1 & 1 & 1 \\
8 & & & & & & & & & 0 & 1 & 1 \\
9 & & & & & & & & & & 0 & 1 \\
10 & & & & & & & & & & & 0
\end{array}$$

Symmetric Matrix

Matrix of shortest Paths through v_0:

$$\sigma(v_0) = \begin{array}{c|ccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
2 & & & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
3 & & & & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
4 & & & & & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
5 & & & & & & 0 & 0 & 0 & 1 & 1 & 0 \\
6 & & & & & & & 0 & 0 & 1 & 1 & 0 \\
7 & & & & & & & & 0 & 1 & 1 & 0 \\
8 & & & & & & & & & 0 & 1 & 1 \\
9 & & & & & & & & & & 0 & 1 \\
10 & & & & & & & & & & & 0
\end{array}$$

Exclude v_0 from its own shortest paths.

Note you can fill any cell for adjacent nodes with zeros.

$$\text{Matrix } B(v_0) = \frac{\sigma(v_0)}{\sigma} \Rightarrow b_{jk} = \frac{\sigma_{jk}(v_0)}{\sigma_{jk}}$$

$$B(v_0) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 0 & & & & & & & & & & \\ & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 1 & 1 & 0 \\ & & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 1 & 1 & 0 \\ & & & 0 & \tfrac{1}{2} & 0 & 0 & 0 & 1 & 1 & 0 \\ & & & & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ & & & & & 0 & 0 & 0 & 1 & 1 & 0 \\ & & & & & & 0 & 0 & 1 & 1 & 0 \\ & & & & & & & 0 & 1 & 1 & 0 \\ & & & & & & & & 0 & 1 & 1 \\ & & & & & & & & & 0 & 1 \\ & & & & & & & & & & 0 \end{bmatrix} \end{array}$$

Again, exclude v_0

Can quickly fill in cells with zero in $\sigma_{jk}(v_0)$

Note these values come only from the top right triangle since we are using an undirected graph

$$B_{twn}(v_0) = \sum_{j,k \neq 0} \frac{\sigma_{jk}(v_0)}{\sigma_{jk}} = \sum_{j,k \neq 0} B(v_0) = 17(1) + 5\left(\tfrac{1}{2}\right) = 19.5$$

Normalize $B_{twn}(v_0)$ w/ $\dfrac{2}{(n-1)(n-2)}$, or all pairs excluding v_0

$$\overline{B_{twn}(v_0)} = \frac{2 \cdot 19.5}{10 \cdot 9} = 19.5/45 = \boxed{0.4333}$$

Repeating Betweenness Centrality for v_5:

$$\sigma(v_5) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 0 & 0 & & 1 & 1 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 1 & & 1 & 1 & 0 & 0 & 1 \\ & & 0 & 1 & 1 & & 1 & 1 & 0 & 0 & 1 \\ & & & 0 & 1 & & 1 & 1 & 0 & 0 & 1 \\ & & & & 0 & & 1 & 1 & 0 & 0 & 1 \\ & & & & & 0 & & & & & \\ & & & & & & 0 & 0 & 1 & 1 & 0 \\ & & & & & & & 0 & 1 & 1 & 0 \\ & & & & & & & & 0 & 0 & 1 \\ & & & & & & & & & 0 & 1 \\ & & & & & & & & & & 0 \end{array}\right] \end{array}$$

Exclude v_5

$$B(v_5) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 0 & 0 & & 1 & 1 & 0 & 0 & 1 \\ & 0 & 0 & 1/2 & 1/2 & & 1 & 1 & 0 & 0 & 1 \\ & & 0 & 1/2 & 1/2 & & 1 & 1 & 0 & 0 & 1 \\ & & & 0 & 1/2 & & 1 & 1 & 0 & 0 & 1 \\ & & & & 0 & & 1 & 1 & 0 & 0 & 1 \\ & & & & & 0 & & & & & \\ & & & & & & 0 & 0 & 1 & 1 & 0 \\ & & & & & & & 0 & 1 & 1 & 0 \\ & & & & & & & & 0 & 0 & 1 \\ & & & & & & & & & 0 & 1 \\ & & & & & & & & & & 0 \end{array}\right] \end{array}$$

$$\text{Btwn}(v_5) = \sum_{j,k\neq 0} \frac{\sigma_{jk}(v_5)}{\sigma_{jk}} = \sum_{j,k\neq 0} B(v_5) = 21(1) + 5(1/2)$$
$$= 21 + 2.5 = 23.5$$

$$\text{Normalized Btwn}(v_5) = \frac{2 \cdot 23.5}{10 \cdot 9} = \frac{23.5}{45} = \boxed{0.5222}$$

Vertex v_5 has the highest betweenness centrality.

Below is a table that includes centrality scores for each node in the graph.

Normalized Centrality Scores

Id	Betweenness	Closeness	Degree Centrality
0	0.433333	0.625	0.7
5	0.522222	0.666667	0.6
6	0.355556	0.5	0.2
1	0	0.5	0.3
2	0	0.5	0.3
3	0	0.47619	0.2
4	0	0.47619	0.2
7	0.2	0.37037	0.2
8	0	0.4	0.1
9	0	0.4	0.1
10	0	0.277778	0.1